KAWAMATA-VIEHWEG VANISHING AND QUINT-CANONICAL MAP OF A COMPLEX THREEFOLD

Meng Chen*

Department of Applied Mathematics
Tongji University, Shanghai, 200092, China

e-mail: chenmb@online.sh.cn

Introduction

Given a complex nonsingular minimal threefold X of general type, Benveniste ([2]) proved that m-canonical map ϕ_m is a birational map onto its image when $m \geq 8$, Matsuki ([15]) showed the same statement for 7-canonical map. In [5], we proved the birationality of 6-canonical map. In [14], Lee proved, independently, that m-canonical map is a birational morphism for $m \geq 6$. Furthermore, the 5-canonical map is birational when $K_X^3 > 2$ according to Ein-Lazarsfeld-Lee. The aim of this note is to prove the following two theorems by a different method:

Theorem 1. *Let X be a complex nonsingular projective threefold with nef and big canonical divisor K_X. Then*

(1) ϕ_5 is a birational map onto its image when $p_g(X) \geq 3$;

(2) if $p_g(X) = 2$ and ϕ_5 is not a birational map, then ϕ_5 is generically finite of degree 2 and $q(X) = h^2(\mathcal{O}_X) = 0$ and $|K_X|$ is composed of a rational pencil of surfaces of general type with $(K^2, p_g) = (1, 2)$.

Theorem 2. *Let X be a complex nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $p_g(X) \leq 1$ and $|2K_X|$ be composed of a pencil of surfaces, i.e., $\dim\phi_2(X) = 1$, then ϕ_5 is a birational map onto its image.*

We would like to put a conjecture here:

The author is partially supported by the National Natural Science Foundation of China

Conjecture. *There exists the exception and the only possible exception to the birationality of 5-canonical map of a complex nonsingular minimal threefold X is one with*

$$(K_X^3, p_g(X), q(X), h^2(\mathcal{O}_X)) = (2, 2, 0, 0).$$

1. A lemma on a surface with $K^2 = 1$ and $p_g = 2$

Let S, with minimal model S_0, be a nonsingular algebraic surface of general type with $K_{S_0}^2 = 1$ and $p_g(S) = 2$. It is well-known that ϕ_5 is birational and ϕ_4 is generically finite of degree 2. In order to make preparation for the proof of our main theorems. We would like to formulate a remark to this kind of surfaces.

Kawamata-Viehweg vanishing theorem will be used throughout this paper in the following form:

Vanishing Theorem. *Let X be a nonsingular complete variety, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:*

(1) D is nef and big;

(2) the fractional part of D has the support with only normal crossings.

Then $H^i(X, \mathcal{O}_X(\lceil D \rceil + K_X)) = 0$ for $i > 0$, where $\lceil D \rceil$ is the minimum integral divisor with $\lceil D \rceil - D \geq 0$.

Remark 1.1. In the case of surfaces, Sakai proved that the Kawamata-Viehweg vanishing holds without the assumption of normal crossings.

Lemma 1.1. *Let S, with minimal model S_0, be a nonsingular projective algebraic surface of general type with $K_{S_0}^2 = 1$ and $p_g(S) = 2$. If $\pi : S \longrightarrow S_0$ is the contraction map, then*

$$\phi_{4.5} := \Phi_{|K_S + 3\pi^*(K_{S_0}) + \lceil \frac{\pi^*(K_{S_0})}{2} \rceil|}$$

is a birational map onto its image.

Proof. If $\pi^*(K_{S_0})$ is an irreducible effective divisor, the lemma is obviously true. Otherwise, we have an effective irreducible divisor D_0 and an effective divisor E_0 such that $D_0 + E_0 \in |\pi^*(K_{S_0})|$ and $D_0 \cdot \pi^*(K_{S_0}) = 1$.

We know that $|K_{S_0}|$ has exactly one base point and has no fixed part. (one may consult (8.1) at page 225 of [1]). A general member $C \in |K_{S_0}|$ is a nonsingular curve of genus 2. Let P be the base point of $|\pi^*(K_{S_0})|$. It is obvious that $\Phi_{|K_S + 3\pi^*(K_{S_0}) + \lceil \frac{D_0 + E_0}{2} \rceil|}$ can separate two general members of $|\pi^*(K_{S_0})|$. We may suppose S be like one of the following three cases without losing of generality:

(1) the exceptional divisors of π do not lie over P;

(2) S is just obtained by blowing up the base point P from a surface like case (1);

(3) S is obtained by several blow ups from a surface like case (2).

Case (1). In this case, there is no changes around P. So we again denote $\pi^{-1}(P)$ by P with no confusion. Let $\tilde{C}$ be the strict transforms of C. Denote $\overline{D_0} = \pi_* D_0$ and $\overline{E_0} = \pi_*(E_0)$.

Let $K_S = \pi^*(K_{S_0}) + \sum E_j$. Note that $\pi|_{\tilde{C}} : \tilde{C} \longrightarrow C$ is an isomorphism. Because $3\pi^*(K_{S_0}) + \frac{D_0+E_0}{2} - \tilde{C} \sim_{\text{num}} \frac{5}{2}\pi^*(K_{S_0})$ is nef and big, therefore, by Vanishing Theorem, we have

$$H^1(S, K_S + 3\pi^*(K_{S_0}) + \lceil \frac{D_0 + E_0}{2} \rceil - \tilde{C}) = 0.$$

Note that, in this case, $K_S|_{\tilde{C}} = \pi^*(K_{S_0})|_{\tilde{C}}$ and $\tilde{C} \in |\pi^*(K_{S_0})|$. We see that

$$\Phi_{|K_S+3\pi^*(K_{S_0})+\lceil \frac{D_0+E_0}{2} \rceil|}|_{\tilde{C}} = \Phi_{|2K_{\tilde{C}}+q|},$$

where $q := D_0|_{\tilde{C}}$ is a point on $\tilde{C}$. Because $\deg(2K_{\tilde{C}} + q) = 5$ and then $2K_{\tilde{C}} + q$ is very ample,

$$\Phi_{|K_S+3\pi^*(K_{S_0})+\lceil \frac{D_0+E_0}{2} \rceil|}$$

is a birational map onto its image.

Case (2). In this case, let S_1 be a surface as case (1) and $\pi_1 : S_1 \longrightarrow S_0$ be the contraction map onto S_0. Let $\pi_2 : S \longrightarrow S_1$ be the blowing up at P, i.e., the base point of $\pi_1^*(K_{S_0})$. Let $C \in |\pi_1^*(K_{S_0})|$ be a general member and $\tilde{C}$ the strict transform of C. Let $D_1 := \pi_{1*} D_0$ and E be the (-1)-curve over P. Denote $\pi := \pi_1 \circ \pi_2$.

We have

$$\begin{aligned} K_S &= \pi_2^*(K_{S_1}) + E \\ &= \pi_2^*(\pi_1^*(K_{S_0}) + \sum E_k) + E \\ &= \pi^*(K_{S_0}) + \pi_2^*(\sum E_k) + E. \end{aligned}$$

We also have that $\pi^*(K_{S_0}) \sim_{\text{lin}} \tilde{C} + E$.

Now we consider the system

$$|K_S + 2\pi^*(K_{S_0}) + \tilde{C} + \lceil \frac{D_0 + E_0}{2} \rceil|.$$

Because

$$K_S + 2\pi^*(K_{S_0}) + \tilde{C} + \lceil \frac{D_0 + E_0}{2} \rceil \leq K_S + 3\pi^*(K_{S_0}) + \lceil \frac{D_0 + E_0}{2} \rceil,$$

we only have to verify the birationality of

$$\Phi_{|K_S+2\pi^*(K_{S_0})+\tilde{C}+\lceil \frac{D_0+E_0}{2} \rceil|}.$$

Because $2\pi^*(K_{S_0}) + \frac{D_0+E_0}{2}$ is nef and big, we have

$$H^1(S, K_S + 2\pi^*(K_{S_0}) + \lceil \frac{D_0 + E_0}{2} \rceil) = 0$$

by Vanishing Theorem. Note that $E_0 = E + E'$, $E' \geq 0$ and $2E \not\leq E_0$. Therefore

$$\Phi_{|K_S+2\pi^*(K_{S_0})+\tilde{C}+\lceil \frac{D_0+E_0}{2} \rceil|}|_{\tilde{C}} = \Phi_{|2K_{\tilde{C}}+q|},$$

where $q = E|_{\tilde{C}}$. $\Phi_{|2K_S+q|}$ is an embedding, because $\deg(2K_S + q) = 5$. Thus

$$\Phi_{|K_S+3\pi^*(K_{S_0})+\lceil \frac{D_0+E_0}{2} \rceil|}$$

is birational.

Case (3). one can easily go through the proof by a similar argument as that of case (2). □

2. Proof of theorem 1

Basic formula. Let X be a nonsingular projective threefold. For a divisor $D \in \mathrm{Div}(X)$, we have

$$\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$$

by Riemann-Roch theorem. A calculation shows that

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbb{Z},$$

therefore $K_X \cdot D^2$ is an even integer, especially K_X^3 is even. If K_X is nef and big, then we obtain by Kawamata-Viehweg's vanishing theorem that

$$p(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)[n(n-1)K_X^3/12 - \chi(\mathcal{O}_X)], \tag{2.1}$$

for $n \geq 2$. Miyaoka ([16]) showed that $3c_2(X) - c_1(X)^2$ is pseudo-effective, therefore we get $K_X^3 \leq -72\chi(\mathcal{O}_X)$ by the Riemann-Roch equality, $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24$. In particular, $\chi(\mathcal{O}_X) < 0$.

Let $f : X \to C$ be a fibration onto a nonsingular curve C. From the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_*\omega_X) \Longrightarrow E^n := H^n(X, \omega_X),$$

a direct calculation shows that

$$h^2(\mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_*\omega_X), \tag{2.2}$$

$$q(X) := h^1(\mathcal{O}_X) = b + h^1(C, R^1 f_*\omega_X). \tag{2.3}$$

Therefore we obtain

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_F)\chi(\mathcal{O}_C) + \Delta_2 - \Delta_1, \tag{2.4}$$

where we set $\Delta_1 := \deg f_*\omega_{X/C}$ and $\Delta_2 := \deg R^1 f_*\omega_{X/C}$. Theorem 1 of [11] tells that $\Delta_1 \geq 0$. Lemma 2.5 of [17] says that $\Delta_2 \geq 0$.

Lemma 2.1. *Let S be a nonsingular algebraic surface, L a nef and big divisor on S. Then*

(1) $\Phi_{|K_S+mL|}$ is a birational map onto its image for $m \geq 4$;
(2) $\Phi_{|K_S+3L|}$ is a birational map onto its image when $L^2 \geq 2$.

Proof. This is a direct result of Corollary 2 of [18]. $\square$

Lemma 2.2. *(See Lemma 2 of* [19]*) Let X be a nonsingular projective variety, D a divisor with $|D| \neq \emptyset$. If the complete linear system $|M|$ is base point free and* $\dim\Phi_{|M|}(X) \geq 2$ *and $\Phi_{|M+D|}$ is not a birational map onto its image, then $\Phi_{|M+D|}|_S$ is also not birational for a general member $S \in |M|$.*

Definition 2.1. Let X be a nonsingular projective threefold. If $\dim \phi_1(X) \geq 2$ and set $K_X \sim_{\text{lin}} M_1 + Z_1$, where M_1 is the moving part and Z_1 the fixed one. We define $\delta_1(X) := K_X^2 \cdot M_1$.

Proposition 2.1. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose* $\dim \phi_1(X) \geq 2$*, then $\delta_1(X) \geq 2$.*

Proof. Let $f_1 : X' \longrightarrow X$ be a succession of blowing-ups with nonsingular centers according to Hironaka such that $g_1 := \phi_1 \circ f_1$ is a morphism. Let $g_1 : X' \xrightarrow{h_2} W_1' \xrightarrow{s_1} W_1 \subset \mathbb{P}^{p_g(X)-1}$ be the Stein factorization of g_1. Let H_1 be a hyperplane section of $W_1 = \overline{\phi_1(X)}$ in $\mathbb{P}^{p_g(X)-1}$ and S_1 be a general member of $|g_1^*(H_1)|$. Since $\dim W_1 \geq 2$, S_1 is a nonsingular irreducible projective surface by Bertini Theorem. Set $f_1^*(M_1) \sim_{\text{lin}} S_1 + E_1'$, $K_{X'} \sim_{\text{lin}} f_1^*(K_X) + E_1$, where E_1 is the ramification divisor for f_1, E_1' is the exceptional divisor for f_1. We have the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "h_1"] \arrow[d, equal] & W_1' \arrow[d, "s_1"] \\
X' \arrow[r, "g_1"'] \arrow[d, "f_1"'] & W_1 \\
X &
\end{tikzcd}$$

We have $\delta_1(X) = K_X^2 \cdot M_1 = f_1^*(K_X)^2 \cdot S_1$. Multiplying $K_X \sim_{\text{lin}} M_1 + Z_1$ by $K_X \cdot M_1$, we have

$$K_X^2 \cdot M_1 = K_X \cdot M_1^2 + K_X \cdot M_1 \cdot Z_1.$$

Since $|S_1|$ is not composed of a pencil, $f_1^*(K_X)$ is nef and big and since S_1 is nef, we have $f_1^*(K_X) \cdot S_1^2 \geq 1$. So that

$$\begin{aligned} K_X \cdot M_1^2 &= f_1^*(K_X) \cdot f_1^*(M_1)^2 = f_1^*(K_X) \cdot f_1^*(M_1) \cdot S_1 \\ &= f_1^*(K_X) \cdot S_1^2 + f_1^*(K_X) \cdot S_1 \cdot E_1' \geq 1. \end{aligned}$$

Whereas, $K_X \cdot M_1^2$ is even and $K_X \cdot M_1 \cdot Z_1 \geq 0$ because $M_1 \cdot Z_1 \geq 0$ as a 1-cycle. Thus we have $K_X^2 \cdot M_1 \geq 2$. $\square$

THEOREM 2.1. *Let X be a nonsingular projective complex threefold with nef and big canonical divisor K_X. Suppose $p_g(X) \geq 3$ and $|K_X|$ be not composed of a pencil of surfaces, i.e.,* $\dim \phi_1(X) \geq 2$, *then ϕ_5 is a birational map onto its image.*

PROOF. We use the same diagram as in the proof of Proposition 2.1 and keep the same notations there. Assume ϕ_5 be not birational, because

$$5K_{X'} \sim_{\text{lin}} \{K_{X'} + 3f_1^*(K_X) + S_1\} + 4E_1 + f_1^*(Z_1) + E_1',$$

$\Phi_{|K_{X'}+3f_1^*(K_X)+S_1|}$ is also not birational. Therefore $\Phi_{|K_{X'}+3f_1^*(K_X)+S_1|}|_{S_1}$ is not birational by Lemma 2.2.

On the other hand, we have $H^1(X', K_{X'} + 3f_1^*(K_X)) = 0$ according to Vanishing Theorem. Thus

$$\Phi_{|K_{X'}+3f_1^*(K_X)+S_1|}|_{S_1} = \Phi_{|K_{S_1}+3L_1|},$$

where we set $L_1 := f_1^*(K_X)|_{S_1}$, which is nef and big and $L_1^2 = \delta_1(X) \geq 2$. Therefore the latter is birational onto its image by Lemma 2.1. Which is a contradiction. $\square$

In the next, we always suppose that $|K_X|$ be composed of a pencil of surfaces. We again use the same diagram as in the proof of Proposition 2.1. Note that W_1' is a nonsingular curve. We usually call h_1 a derived fibration of ϕ_1. Let F be a general fiber of h_1. Then F must be a nonsingular projective surface of general type by Bertini Theorem. Denote $b := g(W_1')$, the geometric genus of curve W_1'.

We can set $g_1^*(H_1) \sim_{\text{num}} aF$, where $a \geq p_g(X) - 1$. Let $\overline{F} := f_{1*}(F)$, then $M_1 \sim_{\text{num}} a\overline{F}$. We will formulate our proof through two steps: (1) $K_X \cdot \overline{F}^2 > 0$ and (2) $K_X \cdot \overline{F}^2 = 0$.

THEOREM 2.2. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $p_g(X) \geq 2$ and $|K_X|$ be composed of a pencil, keeping the above notations, if $K_X \cdot \overline{F}^2 > 0$, then ϕ_5 is a birational map onto its image.*

PROOF. We have

$$5K_{X'} \sim_{\text{lin}} \{K_{X'} + 3f_1^*(K_X) + aF\} + 4E_1 + f_1^*(Z_1) + E_1'.$$

Consider the system $|K_{X'} + 3f_1^*(K_X) + aF|$, we have $H^1(X', K_{X'} + 3f_1^*(K_X)) = 0$. Generically, we can take $g_1^*(H_1)$ be a disjoint union of fibers F_i $(1 \leq i \leq a)$. Therefore we have the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X'}(K_{X'} + 3f_1^*(K_X)) &\longrightarrow \mathcal{O}_{X'}(K_{X'} + 3f_1^*(K_X) + g_1^*(H_1)) \\ &\longrightarrow \oplus_{i=1}^a \mathcal{O}_{F_i}(K_{F_i} + 3L_i) \longrightarrow 0, \end{aligned}$$

where $L_i = f_1^*(K_X)|_{F_i}$, which is nef and big and

$$L_i^2 = K_X^2 \cdot \overline{F} \geq K_X \cdot \overline{F}^2 \geq 2. \ (K_X \cdot \overline{F}^2 \text{ is even})$$

From the above exact sequence, we see that

$$\Phi_{|K_{X'}+3f_1^*(K_X)+g_1^*(H_1)|}|_{F_i} = \Phi_{|K_{F_i}+3L_i|}$$

is a birational map onto its image by Lemma 2.1. Thus ϕ_5 is birational. $\square$

Lemma 2.3. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Keeping the above notations, if $K_X \cdot \overline{F}^2 = 0$, then*

$$\mathcal{O}_F(f_1^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_{F_0})).$$

Proof. This can be obtained by a similar argument to that for *Case* β) of (*ii*), Theorem 7 of [15]. $\square$

Theorem 2.3. *Under the same assumption as in Lemma 2.3, If the minimal model F_0 of F is not a surface with $K_{F_0}^2 = 1$ and $p_g(F_0) = 2$, then ϕ_5 is a birational map onto its image.*

Proof. The proof is almost the same as that of Theorem 2.2. The only difference occurs on $L_i = f_1^*(K_X)|_{F_i}$. From Lemma 2.3, we see that $L_i \sim_{\text{lin}} \pi^*(K_{F_0})$ and therefore $\Phi_{|K_{F_i}+3L_i|} = \Phi_{|4K_{F_i}|}$ is birational under the assumption of this theorem. $\square$

Theorem 2.4. *Under the same assumption as in Lemma 2.3, if the minimal model F_0 of F is just the surface with $K_{F_0}^2 = 1$ and $p_g(F_0) = 2$, then ϕ_5 is also a birational map in one of the following two cases:*

(1) $p_g(X) \geq 3$;

(2) $p_g(X) = 2$ and $b := g(W_1') \neq 0$.

Proof. In the two cases of this theorem, we can see that $a \geq 2$. Fix an effective divisor $K_0 \in |K_X|$. Actually, we can modify f_1 such that

$$f_1^*(K_0) = \sum_{i=1}^{a} F_i + E_1' + f_1^*(Z_1)$$

has support with only normal crossings. Thus, from now on, we always suppose f_1 has this property. For a general fiber F of h_1, We have $g_1^*(H_1) \sim_{\text{num}} 2F + \sum_{i=1}^{a-2} F_i$. For $\mathbb{Q}$-divisor

$$\overline{G} := 4f_1^*(K_X) - F - \frac{1}{2}(F_1 + \cdots + F_{a-2} + E_1' + f_1^*(Z_1)),$$

it is nef and big. Denote

$$G := [\frac{F_1 + \cdots + F_{a-2} + E_1' + f_1^*(Z_1)}{2}],$$

then $H^1(X', K_{X'} + 4f_1^*(K_X) - F - G) = 0$ by Vanishing Theorem. Considering the system $|K_{X'} + 4f_1^*(K_X) - G|$, it is obvious that

$$K_{X'} + 4f_1^*(K_X) - G \leq 5K_{X'}.$$

In order to proof the birationality of ϕ_5, we only have to verify for

$$\Phi_{|K_{X'}+4f_1^*(K_X)-G|}.$$

From the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X'}(K_{X'} + 4f_1^*(K_X) - G - F) \longrightarrow \mathcal{O}_{X'}(K_{X'} + 4f_1^*(K_X) - G) \\ \longrightarrow \mathcal{O}_F(K_F + 3f_1^*(K_X)|_F + \lceil \frac{E_1'|_F + f_1^*(Z_1)|_F}{2} \rceil) \longrightarrow 0, \end{aligned}$$

we see that

$$\Phi_{|K_{X'}+4f_1^*(K_X)-G|}|_F = \Phi_{|K_F+3f_1^*(K_X)|_F+\lceil \frac{E_1'|_F+f_1^*(Z_1)|_F}{2} \rceil|}.$$

Note that $f_1^*(K_X)|_F \sim_{\text{lin}} E_1'|_F + f_1^*(Z_1)|_F$. From Lemma 2.3, we have $f_1^*(K_X)|_F \sim_{\text{lin}} \pi^*(K_{F_0})$, where $\pi : F \longrightarrow F_0$ is the contraction to the minimal model. Thus we complete the proof by Lemma 1.1. $\square$

Finally, if $p_g(X) = 2$ and $|K_X|$ is composed of a pencil of surfaces, the above method is not effective. But from the proof of Theorem 2.3, we can see that ϕ_5 is at least a generically finite map of degree 2. By formula (2.2) and (2.3), we can easily get $q(X) = h^2(\mathcal{O}_X) = 0$.

Combining the arguments of this section, we obtain Theorem 1.

3. On a bicanonical pencil of surfaces of general type

In order to study the case when $p_g(X) \leq 1$, it is natural to study ϕ_2. This section is a preparation for the proof of Theorem 2.

Let X be a nonsingular minimal projective threefold. If $|2K_X|$ is composed of a pencil of surfaces, i.e., the image of X through $\Phi_{|2K_X|}$ is of dimension 1, we can find a birational modification $f_2 : X' \to X$ such that $g_2 = \Phi_{|2K_X|} \circ f_2$ is a morphism. Let

$W_2 = \overline{\phi_2(X)} \subset \mathbb{P}^{p(2)-1}$, and $g_2 = s_2 \circ h_2$ is a Stein-factorization of g_2. We have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "h_2"] \arrow[d, equal] & C \arrow[d, "s_2"] \\ X' \arrow[r, "g_2"'] \arrow[d, "f_2"'] & W_2 \\ X & \end{tikzcd}$$

where $h_2 : X' \to C$ is called a derived fibration of ϕ_2. Let F be a general fiber of h_2, then F must be a nonsingular projective surface by Bertini Theorem. Denote $b := g(C)$, the genus of C.

LEMMA 3.1. *(Claim 9.1 of [15]) Let X be a nonsingular minimal projective threefold of general type, if $|2K_X|$ is composed of a pencil of surfaces, then*

$$\mathcal{O}_F(f_2^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_{F_0})),$$

where $\pi : F \to F_0$ is the birational contraction onto the minimal model.

LEMMA 3.2. *Under the same assumption as in Lemma 3.1, we have $K_{F_0}^2 \leq 3$ and F is of one of the following two cases:*

(1) $q(F) = 0$, $p_g(F) \leq 3$;
(2) $p_g(F) = q(F) = 1$.

PROOF. Let $f_2^*(2K_X) \sim_{\text{lin}} g_2^*(H_2) + Z_2'$, where Z_2' is the fixed part and H_2 is a general hyperplane section of W_2. Obviously we have $g_2^*(H_2) \sim_{\text{num}} a_2 F$, $a_2 \geq p(2) - 1$. From Lemma 3.1, we have

$$K_{F_0}^2 = (f_2^*(K_X)|_F)^2 = f_2^*(K_X)^2 \cdot F.$$

Let $2K_X \sim_{\text{lin}} M_2 + Z_2$, where M_2 is the moving part and Z_2 is the fixed part. We also have $M_2 = f_{2*}(g_2^*(H_2))$. Denote $\overline{F} = f_{2*}F$, then $M_2 \sim_{\text{num}} a_2 \overline{F}$. By projection formula, one has

$$K_X^2 \cdot \overline{F} = f_2^*(K_X)^2 \cdot F = K_{F_0}^2.$$

Because K_X is nef, we have $2K_X^3 \geq a_2 K_X^2 \cdot \overline{F}$. Therefore

$$K_X^2 \cdot \overline{F} \leq \frac{2}{a_2} K_X^3 \leq \frac{4K_X^3}{K_X^3 - 6\chi(\mathcal{O}_X) - 2} \leq \frac{4K_X^3}{K_X^3 + 4} < 4,$$

and then $K_{F_0}^2 \leq 3$. Because $2p_g(F_0) - 4 \leq K_{F_0}^2$, $p_g(F_0) \leq 3$. If $q(F) > 0$, then Bombieri's theorem([3]) tells that $K_{F_0}^2 \geq 2\chi(\mathcal{O}_{F_0}) \geq 2$, therefore $\chi(\mathcal{O}_{F_0}) = 1$, i.e., $p_g(F_0) = q(F_0)$. By Debarre's result([7]), we have $K_{F_0}^2 \geq 2p_g(F_0)$, therefore $p_g(F_0) = 1$. □

Lemma 3.3. *Under the same assumption as in Lemma 3.1, then $b = 0$ or $b = 1$.*

Proof. Keep the notations above. If $b > 0$, then ϕ_2 is actually a morphism. Thus we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h_2} & C \\ \| & & \downarrow{s_2} \\ X & \xrightarrow[\phi_2]{} & W_2 \end{array}$$

Let $\mathcal{E}_0$ be a saturated subbundle of $f_*(\omega_X^{\otimes 2})$ which is generated by $H^0(C, f_*(\omega_X^{\otimes 2}))$. Because $|2K_X|$ is composed of a pencil and ϕ_2 factors through h_2, $\mathcal{E}_0$ must be a subbundle of rank 1. Let $\mathcal{E} = f_*(\omega_X^{\otimes 2})$, we have the following exact sequence

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{E}_1 \to 0$$

and

$$f_*(\omega_{X/C}^{\otimes 2}) \to \mathcal{E}_1 \otimes \omega_C^{\otimes -2} \to 0.$$

Let $r = rk\mathcal{E} = h^0(2K_F) = K_{F_0}^2 + \chi(\mathcal{O}_{F_0}) \geq 2$. By Kawamata's result ([11]), $f_*(\omega_{X/C}^{\otimes 2})$ is semi-positive. Therefore $\mathcal{E}_1 \otimes \omega_C^{\otimes -2}$, as a quotient, satisfies $\deg(\mathcal{E}_1 \otimes \omega_C^{\otimes -2}) \geq 0$, i.e., $\deg \mathcal{E}_1 \geq 4(r-1)(b-1)$. We have

$$\begin{aligned} h^1(\mathcal{E}_0) \geq h^0(\mathcal{E}_1) &\geq \deg \mathcal{E}_1 + (r-1)(1-b) \\ &\geq 3(r-1)(b-1). \end{aligned}$$

Noting that $deg\mathcal{E}_0 > 0$, if $h^1(\mathcal{E}_0) > 0$, then by Clifford's theorem,

$$\deg \mathcal{E}_0 \geq 2h^0(\mathcal{E}_0) - 2 > h^0(\mathcal{E}_0)$$

where $h^0(\mathcal{E}_0) = p(2)(X) \geq 4$. We have

$$h^1(\mathcal{E}_0) = (h^0(\mathcal{E}_0) - deg\mathcal{E}_0) + (b-1) < b-1$$

thus $3(r-1)(b-1) < b-1$, which is impossible. Therefore $h^1(\mathcal{E}_0) = 0$ and $b = 1$. □

Lemma 3.4. *Under the same assumption as in Lemma 3.1, we have $p_g(F) \geq 1$.*

Proof. If $p_g(F) = 0$, because F is a surface of general type, $q(F) = 0$. Therefore $R^1 h_{2*}\omega_{X'} = 0$. By basic formula, we have $q(X) = q(X') = b$ and $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{X'}) = 0$. If $p_g(X) \geq 1$, we know that $p_g(F) \geq 1$, therefore, under the above assumption, we must have $p_g(X) = 0$. From Lemma 3.3,

$$\chi(\mathcal{O}_X) = 1 - q(X) = 1 - b \geq 0.$$

which is impossible, because $\chi(\mathcal{O}_X) < 0$. □

Theorem 3.1. *Let X be a nonsingular projective minimal threefold of general type, suppose that $|2K_X|$ be composed of a pencil of surfaces, then X must be of one of the following types:*

(1) $q(F)=0$, $1 \le K_{F_0}^2 \le 3$:

(11) $b=1$, $p_g(F)=q(X)=1$, $h^2(\mathcal{O}_X)=0$, $p_g(X)\ge 2$;

(12) $b=1$, $1\le p_g(F)\le 3$, $q(X)=1$, $h^2(\mathcal{O}_X)=0$, $p_g(X)=1$, $\chi(\mathcal{O}_X)=-1$;

(13) $b=0$, $p_g(F)=1$, $q(X)=h^2(\mathcal{O}_X)=0$, $p_g(X)\ge 2$.

(2) $p_g(F)=q(F)=1$, $K_{F_0}^2=2,3$:

(21) $b=1$, $q(X)=2$, $h^2(\mathcal{O}_X)=1$, $p_g(X)\ge 1$;

(22) $b=1$, $q(X)=1$, $h^2(\mathcal{O}_X)=0$, $p_g(X)=1$;

(23) $b=1$, $q(X)=1$, $p_g(X)\ge 2$;

(24) $b=0$, $q(X)=1$, $h^2(\mathcal{O}_X)=0$, $p_g(X)\ge 1$;

(25) $b=0$, $q(X)=0$, $p_g(X)\ge 2$.

Proof. From Lemma 3.2, we know that F is of two cases: (1) $q(F)=0$; (2) $p_g(F)=q(F)=1$.

Case (1):

We have $\triangle_2 = degR^1h_{2*}\omega_{X'/C}=0$, therefore $q(X)=b$ and

$$h^2(\mathcal{O}_X)=h^1(h_{2*}\omega_{X'}).$$

Case$(1)^1$: $p_g(X)\ge 2$. It is obvious that $|K_X|$ is composed of a pencil of surfaces and ϕ_1 generically factors through ϕ_2. Take a common birational modification $f: X'\to X$ such that $g_i=\phi_i\circ f$ $(i=1,2)$ is a morphism. We have the following commutative diagram:

$$\begin{array}{ccc}
X' & \xrightarrow{h} & C \\
\| & & \downarrow{\scriptstyle s_2} \\
X' & \xrightarrow[g_2]{} & W_2 \\
\| & & \downarrow{\scriptstyle s_1} \\
X' & \xrightarrow[g_1]{} & W_1 \\
{\scriptstyle f}\downarrow & & \\
X & &
\end{array}$$

Let $g_2 := s_2\circ h$ is a Stein-factorization of g_2, then $g_1=(s_1\circ s_2)\circ h$ is a Stein-factorization of g_1. Let H_1, H_2 be the general hyperplane section of W_1, W_2, respectively. We have $g_1^*(H_1)\sim_{\text{lin}}\sum_{i=1}^{a_1}F_i$, F_i is a fiber of h for every i and $a_1\ge p_g(X')-1$.

If $b=1$, then ϕ_1, ϕ_2 are morphisms. We may suppose that $X=X'$. We also have $q(X)=1$. Using a similar method to that in the proof of Lemma 3.3, one has $h^1(h_*\omega_X)=0$, therefore $h^2(\mathcal{O}_X)=0$. Upon an open Zariski subset of C, we have the following exact sequence:

$$0\to\mathcal{O}_{X'}(K_{X'})\to\mathcal{O}_{X'}(K_{X'}+g_1^*(H_1))\to\oplus_{i=1}^{a_1}\mathcal{O}_{F_i}(K_{F_i})\to 0. \tag{3.1}$$

We have the surjective map

$$H^0(K_{X'}+g_1^*(H_1))\to\oplus_{i=1}^{a_1}H^0(K_{F_i}),$$

thus $p_g(F)=1$, otherwise because

$$\Phi_{|K_{X'}+g_1^*(H_1)|}|_{F_i}=\Phi_{|K_{F_i}|},$$

$\dim\phi_2(X)\geq 2$, a contraction to our assumption. Thus X corresponds to type (11) of the Theorem.

If $b=0$, then $q(X)=0$. Because $\chi(\mathcal{O}_X)=1+h^2(\mathcal{O}_X)-p_g(X)<0$,

$$h^2(\mathcal{O}_X)\leq p_g(X)-2. \tag{3.2}$$

Noting that $|K_{X'}+g_1^*(H_1)|$ is also composed of a pencil of surfaces, we can easily see that $h^0(K_{X'}+g_1^*(H_1))=2p_g(X')-1=2p_g(X)-1$. $g_1^*(H_1)\sim_{\text{num}}a_1F$, where $a_1=p_g(X')-1$. From (3.1) and (3.2), we obtain

$$a_1p_g(F)\leq p_g(X)-1+h^2(\mathcal{O}_X)\leq 2p_g(X)-3$$

i.e., $(p_g(X)-1)p_g(F)\leq 2p_g(X)-3$. Therefore $p_g(F)=1$, and then $h_*\omega_{X'}$ is a rank one vector bundle. Because $\deg h_*\omega_{X'}>0$, $h^2(\mathcal{O}_X)=h^1(h_*\omega_{X'})=0$. Therefore X corresponds to type (13).

Case$(1)^2$: $p_g(X)\leq 1$. From $\chi(\mathcal{O}_X)=1-q(X)+h^2(\mathcal{O}_X)-p_g(X)<0$, we get $q(X)>0$ and then $b=q(X)=1$, $h^2(\mathcal{O}_X)=0$, $p_g(X)=1$, $\chi(\mathcal{O}_X)=-1$. X corresponds to type (12).

Case (2):

In this case, $R^1h_*\omega_{X'}$ is a rank one vector bundle. Because $R^1h_*\omega_{X'/C}$ is semi-positive, $h^1(R^1h_*\omega_{X'})\leq 1$. Note that $h_*\omega_{X'}$ is also a rank one vector bundle and $b=0,1$. From Riemann-Roch, we have $h^1(h_*\omega_{X'})=0$ if $p_g(X)\geq 2$.

Case$(2)^1$: $p_g(X)\geq 2$. If $h^1(R^1h_*\omega_{X'})=1$, then $R^1h_*\omega_{X'}\cong\omega_C$. When $b=1$, then $q(X)=2$, $h^2(\mathcal{O}_X)=1$. X corresponds to type (21); when $b=0$, then $q(X)=1$, $h^2(\mathcal{O}_X)=0$. X corresponds to type (24).

If $h^1(R^1h_*\omega_{X'}) = 0$, then $q(X) = b$. When $b = 1$, X corresponds to type (23); when $b = 0$, X corresponds to type (25).

Case$(2)^2$: $p_g(X) \leq 1$. From $\chi(\mathcal{O}_X) < 0$, we get $q(X) > 0$. $q(X) = b + h^1(R^1h_*\omega_{X'})$. When $b = 0$, then $h^1(R^1h_*\omega_{X'}) = 1$, $R^1h_*\omega_{X'} \cong \omega_C$. In this case, $q(X) = p_g(X) = 1$ and $h^2(\mathcal{O}_X) = 0$, $\chi(\mathcal{O}_X) = -1$, X corresponds to type (24). When $b = 1$, then there is only two possibilities, i.e., $(q(X), h^2(\mathcal{O}_X), p_g(X)) = (2, 1, 1)$ and $(1, 0, 1)$. The former corresponds to type (21), the latter to type (22). $\square$

Corollary 3.1. *Let X be a nonsingular minimal projective threefold of general type, if $|2K_X|$ is composed of a pencil of surfaces, then $q(X) \leq 2$ and $p_g(X) \geq 1$.*

4. Proof of theorem 2

In this section, we mainly discuss the case when $p_g(X) \leq 1$ and always suppose $|2K_X|$ be composed of a pencil of surfaces. From Theorem 3.1, we see that X corresponds to type (12), type (21), type (22) and type (24). We keep the same notations and use the first commutative diagram of the former section.

Theorem 4.1. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $|2K_X|$ be composed of a pencil of surfaces, X not corresponding to type (12), then ϕ_5 is a birational map onto its image.*

Proof. Considering the system $|K_{X'} + 2f_2^*(K_X) + g_2^*(H_2)|$, we can take a standard argument to this situation. Simply, we get from Lemma 3.1 that, for a general fiber F of h_2,

$$\Phi_{|K_{X'}+2f_2^*(K_X)+g_2^*(H_2)|}|_F = \Phi_{|K_F+2\pi^*(K_{F_0})|} = \Phi_{|3K_F|}.$$

The only exception to the birationality of the 5-canonical map for a minimal surface F_0 is one with

$$(K_{F_0}^2, p_g(F_0)) = (1, 2) \quad \text{or} \quad (2, 3).$$

Which just corresponds to type (12). $\square$

Theorem 4.2. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $|2K_X|$ be composed of a pencil of surfaces and X corresponding to type (12), then ϕ_5 is also a birational map onto its image.*

Proof. Using the first commutative diagram in §3, we have $f_2^*(2K_X) \sim_{\text{lin}} g_2^*(H_2) + Z_2'$, where Z_2' is the fixed part. Take some hyperplane section $\overline{H_2}$ such that $g_2^*(\overline{H_2}) = \sum_{i=1}^{a_2} F_i$, where $a_2 = p(2) \geq 4$ noting that X corresponds to type (12). At first, we can modify f_2 such that $\sum_{i=1}^{a_2} F_i + Z_2'$ has support with only normal crossings.

Let $D \in |f_2^*(K_X)|$ be the unique effective divisor. Because $2D \sim_{\text{lin}} 2f_2^*(K_X)$, there is a hyperplane section H_2^0 of W_2 in $\mathbb{P}^{p(2)-1}$ such that $2D = g_2^*(H_2^0) + Z_2'$. Set

$Z'_2 := Z_V + 2Z_H$, where Z_V is the vertical part with respect to fibration $h_2 : X' \to C$ and $2Z_H$ is the horizontal part. Thus

$$D = \frac{1}{2}[g_2^*(H_2^0) + Z_V] + Z_H.$$

Noting that D is a divisor, for a general fiber F, $Z_H|_F = D|_F \sim_{\text{lin}} \pi^*(K_{F_0})$ by lemma 3.1.

Considering the $\mathbb{Q}-$divisor

$$K_{X'} + 4f_2^*(K_X) - F - \frac{1}{4}(F_5 + \cdots + F_{a_2}) - \frac{1}{4}Z_V - \frac{1}{2}Z_H,$$

set

$$G := 4f_2^*(K_X) - \frac{1}{4}(F_5 + \cdots + F_{a_2}) - \frac{1}{4}Z_V - \frac{1}{2}Z_H$$

and

$$D_0 := \lceil G \rceil = 3f_2^*(K_X) + \lceil \frac{Z_H}{2} \rceil - \text{vertical divisors.}$$

For a general fiber F, $G - F \sim_{\text{num}} \frac{7}{2}f_2^*(K_X)$ is nef and big. Therefore, by vanishing theorem, $H^1(X', K_{X'} + D_0 - F) = 0$. We then have the surjective map

$$H^0(X', K_{X'} + D_0) \longrightarrow H^0(F, K_F + 3\pi^*(K_{F_0}) + \lceil \frac{\pi^*(K_{F_0})}{2} \rceil).$$

If F is not a surface with $(K^2, p_g) = (1, 2)$, then $\Phi_{|K_F + 3\pi^*(K_{F_0}) + \lceil \frac{\pi^*(K_{F_0})}{2} \rceil|}$ is birational on F. Otherwise, we have the same statement by Lemma 1.1. Therefore $\Phi_{|K_{X'}+D_0|}$ is birational and so is $\Phi_{|5K_{X'}|}$. $\square$

References

[1] Barth, W., Peter, C. and Van de Van, A.: Compact Complex Surface, Berlin Heidelberg New York Tokyo: Springer 1980.

[2] Benveniste, X.: Sur les applications pluricanoniques des variétés de type très général en dimension 3, *Amer. J. Math.* **108** (1986), 433–449.

[3] Bombieri, E.: Canonical models of surfaces of general type, *Publ. Math. Inst. Hautes Etude. Sci.* **42**, 171-219(1973).

[4] Chen, M.: An extension of Benveniste-Matsuki's method on 6-canonical maps for threefolds, *Comm. in Algebra* **22**, 5759–5767(1994).

[5] Chen, M.: On pluricanonical maps for threefolds of general type, *J. Math. Soc. Japan*, Vol.**50**, No.4(1998), to appear.

[6] Chen, M.: Complex varieties of general type whose canonical systems are composed with pencils, *J. Math. Soc. Japan*, Vol.**51**, No.2(1999), to appear.

[7] Debarre, O.: Addendum, inégalitiés numériques pour les surfaces de type général, *Bull. Soc. Math. France* **111**/4, 301-302(1983).

[8] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math.* **79**, 109–326(1964).

[9] Kawamata, Y.: Cone of curves of algebraic varieties, *Ann. of Math.* **119**, 603–633(1984).

[10] Kawamata, Y.: A generalization of Kodaira-Ramanujam's vanishing theorem, *Math. Ann.*, **261**(1982), 43-46.

[11] Kawamata, Y.: Kodaira dimension of algebraic fiber spaces over curves, *Invent. Math.* **66**, 57-71(1982).

[12] Kollár, J.: Higher direct images of dualizing sheaves I, *Ann. of Math.* **123**, 11–42(1986).

[13] Kollár, J.: Higher direct images of dualizing sheaves II, *Ann. of Math.* **124**, 171–202(1986).

[14] Lee, S.: Remarks on pluricanonical and the adjoint linear series, preprint.

[15] Matsuki, K.: On pluricanonical maps for 3-folds of general type, *J. Math. Soc. Japan* **38**, 339–359(1986).

[16] Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety, In: *Algebraic Geometry*, Sendai, 1985 (Avid. Stud. in Pure Math. **10**, 1987, pp. 449–476).

[17] Ohno, K.: Some inequalities for minimal fibrations of surfaces of general type over curves, *J. Math. Soc. Japan,* **44**(1992), 643-666.

[18] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces, *Ann. of Math.*, **127**(1988), 309-316.

[19] Tankeev, S. G.: On n-dimensional canonically polarized varieties and varieties of fundamental type, *Izv. Akad*, Nauk. SSSR, Ser. Mat., **35**(1971), 31-44; *Math. USSR Izv.*, **5**(1971), 29-43.

[20] Viehweg, E.: Vanishing theorems, *J. reine angew. Math.*, **335**(1982), 1-8.